Understanding Small-Amplitude Oscillatory Shear

Keywords: Small-amplitude oscillatory shear; Storage modulus; Loss modulus; Complex viscosity, Loss tangent

TB112

ABSTRACT

This note explains in detail the testing principles of small-amplitude oscillatory shear (SAOS), clarifies the physical quantities obtained from SAOS tests and their complex representation, and describes how instruments process SAOS data.

SMALL-AMPLITUDE OSCILLATORY TEST

In an oscillation shear experiment, the material is subjected to either a sinusoidal strain (as shown in Figure 1) or a sinusoidal stress and the corresponding response is measured.

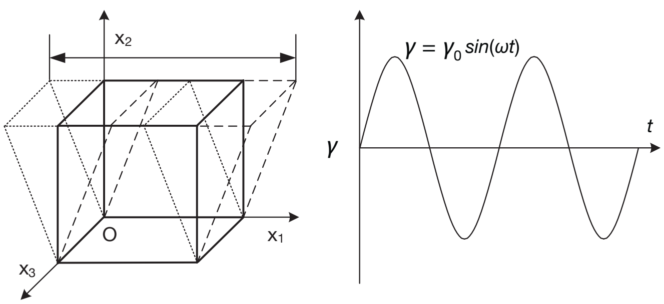

Figure 1. Oscillatory shear deformation

When a sinusoidal deformation or deformation rate is applied, the material's response is the stress. If the amplitude of the applied sinusoidal strain (γ_0) is less than a certain critical value (γ_c), the profile of the responding stress is also sinusoidal. Conversely, when the amplitude of the applied sinusoidal stress (σ_0) is below a certain critical value (σ_c), the profile of the responding strain is also sinusoidal. In this case, the stress amplitude (σ_0) is linearly proportional to the strain amplitude, i.e. $\sigma_0 \propto \gamma_0$, which means the correlation between the response and excitation is totally independent of them. The region of the excitation below its critical value is called the linear regime, and the material's function can be defined in this regime.

During an oscillation test, a sinusoidal strain (as shown in Equation 1) is often applied onto a material:

$$\gamma = \gamma_0 \sin(\omega t) \qquad (1)$$

The corresponding strain rate is:

$$\dot{\gamma} = \frac{d\gamma}{dt} = \omega\gamma_0 \cos(\omega t) \qquad (2)$$

The elastic response of the material follows Hooke's law and the elastic stress component (σ') can be expressed as Equation 3:

$$\sigma' = G'\gamma_0 \sin(\omega t) \qquad (3)$$

where G' is the dynamic elastic modulus.

The viscous response of the material follows Newton's law and the viscous stress component (σ'') can be expressed as Equation 4:

$$\sigma'' = \eta'\omega\gamma_0 \cos(\omega t) \qquad (4)$$

where η' is the dynamic viscosity.

When the amplitude of the applied sinusoidal strain (γ_0) is less than the critical value (γ_c) and is still in the linear regime, the total stress (σ) can be the sum of the elastic stress component (σ') and the viscous stress component (σ''), as shown in Figure 2.

$$\sigma = \sigma' + \sigma'' = G'\gamma_0 \sin(\omega t) + \eta'\omega\gamma_0 \cos(\omega t) \qquad (5)$$

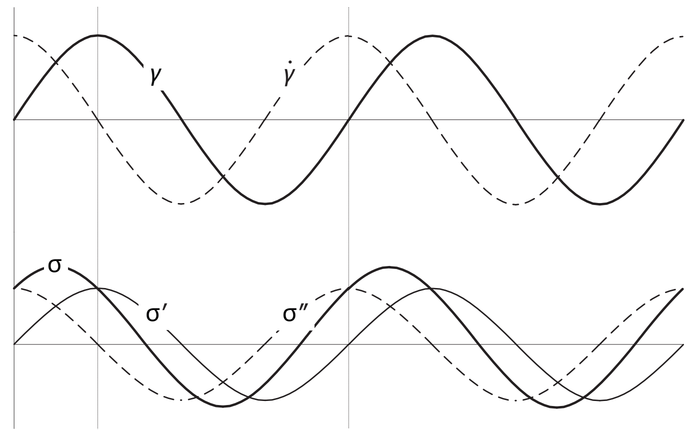

Figure 2. Synthesis of small amplitude sinusoidal oscillation viscoelastic stress

Define the dynamic viscous modulus as G'', and written as:

$$G'' = \eta'\omega \qquad (6)$$

Equation (5) can now be rewritten as Equation 7:

$$\sigma = \sigma' + \sigma'' = G'\gamma_0 \sin(\omega t) + G''\gamma_0 \cos(\omega t) \qquad (7)$$

Since the stress response also shows the sinusoidal characteristics and there is a phase shift (δ) between the stress and the strain, the total stress can be written as:

$$\sigma = \sigma_0 \sin(\omega t + \delta) \qquad (8)$$

where σ_0 is the amplitude of the oscillation stress. Expanding the equation yield Equation 9 below:

$$\sigma = \sigma_0 \cos\delta \sin(\omega t) + \sigma_0 \sin\delta \cos(\omega t) \qquad (9)$$

Setting Equations 7 and 9 equal to each other, G' and G'' can now be written as:

$$G' = (\sigma_0/\gamma_0)\cos\delta,\ G'' = (\sigma_0/\gamma_0)\sin\delta \qquad (10)$$

The Complex Representation of the Small-Amplitude Oscillation

For viscoelastic materials, there is always a phase shift between the response and the excitation, which makes the viscoelastic parameter impossible to directly obtain from the transient stress to strain because the ratio σ/γ is time invariant.

According to Euler's equation (11), the real part and the imaginary part of $e^{i\omega t}$ are separately cosine and sine functions, so it can be considered to use the real part or imaginary part to represent the transient strain or stress.

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \tag{11}$$

In other words, the transient strain and stress can be expressed separately as complex notation:

$$\gamma^* = \gamma_0 e^{i\omega t},\ \sigma^* = \sigma_0 e^{i(\omega t+\delta)} \tag{12}$$

Thus, the complex modulus (G^*) can be obtained from the ratio of the complex stress to the complex strain and the complex modulus is time independent:

$$G^* = \frac{\sigma^*}{\gamma^*} = \frac{\sigma_0 e^{i(\omega t+\delta)}}{\gamma_0 e^{i\omega t}} = \frac{\sigma_0}{\gamma_0} e^{i\delta} = \frac{\sigma_0}{\gamma_0} \cos\delta + i\frac{\sigma_0}{\gamma_0} \sin\delta \tag{13}$$

Comparison of the real and imaginary parts in Equations 13 and 10 show that

$$G^* = G' + iG'' \tag{14}$$

Therefore, Equation 15 can be obtained:

$$|G^*| = \frac{\sigma_0}{\gamma_0} = \sqrt{G'^2 + G''^2} \tag{15}$$

The complex strain rate ($\dot{\gamma}^*$) can be obtained from the derivative of the complex strain (γ^*).

$$\dot{\gamma}^* = \frac{d\gamma^*}{dt} = i\omega\gamma_0 e^{i\omega t} = i\omega\gamma^* \tag{16}$$

The complex viscosity (η^*) can be also obtained as:

$$\eta^* = \frac{\sigma^*}{\dot{\gamma}^*} = \frac{G^*}{i\omega} = \frac{G''}{\omega} - i\frac{G'}{\omega} = \eta' - i\frac{G'}{\omega} \tag{17}$$

Define η'' as the ratio of the storage modulus to the angular frequency:

$$\eta'' = \frac{G'}{\omega} \tag{18}$$

Thus, the complex can be shown as:

$$\eta^* = \eta' - i\eta'' \tag{19}$$

Now, Equation 20 can be obtained:

$$|\eta^*| = \frac{|G^*|}{\omega} = \sqrt{\eta'^2 + \eta''^2} \tag{20}$$

If the real strain and real stress are expressed by the sinusoidal functions, they can be written as:

$$\gamma = Im\{\gamma^*\},\ \sigma = Im\{\sigma^*\} \tag{21}$$

In this case, the relationship between the complex representation of the strain and stress in time domain is shown in Figure 3.

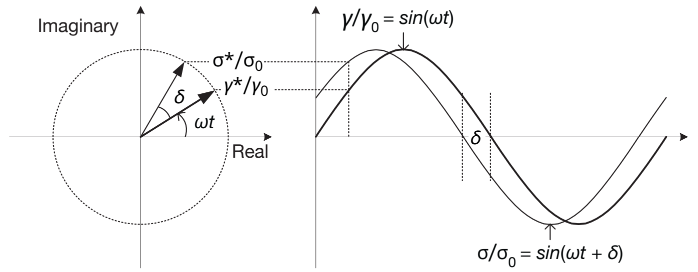

Figure 3. The correspondence between the complex representation of the strain and stress and their time-domain profile

If the real strain and real stress are expressed by the cosine functions, they can be written as:

$$\gamma = Re\{\gamma^*\},\ \sigma = Re\{\sigma^*\} \tag{22}$$

In this case, the relationship between the complex representation of the strain and stress in the time domain is shown in Figure 4.

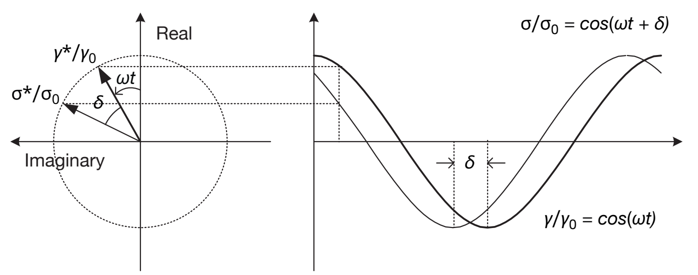

Figure 4. The correspondence between the complex representation of the strain and stress and their time-domain profile

Instrument Processing of Small Amplitude Oscillation Data

In the rheometer, the transient strain and stress signals are digitized and are available to the instrument controller as an array of n measured quantities sampled over an equal time interval Δt, as shown in Table 1.

Table 1. Periodically changing strain and stress values

The time of data point acquisition within the cycle	Transient strain	Transient stress
Δt	γ_1	σ_1
$2 * \Delta t$	γ_2	σ_2
...	...	...
$n * \Delta t$	γ_n	σ_n
...	...	...
$N * \Delta t$	γ_N	σ_N

The graphical visualization of the transient stress and the strain in the above table can be show in Figure 5.

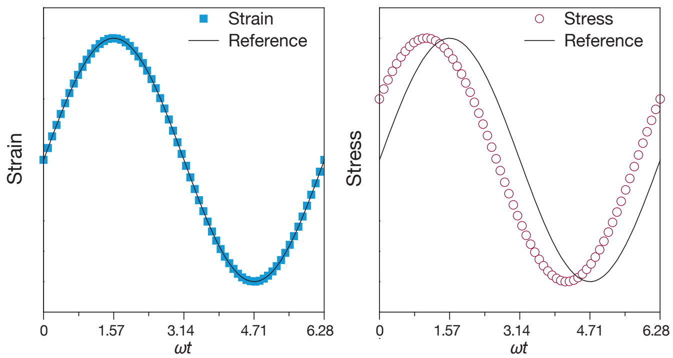

Figure 5. Raw data of oscillation waveform

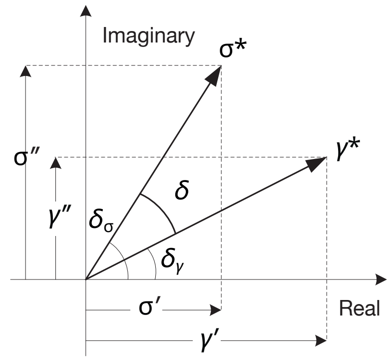

Figure 6. The complex plane representation of strain and stress

The time-domain signals $s(n)$ of the transient strain and stress can be transformed by a discrete Fourier transform (DFT), shown in Equation 23, to its frequency domain spectrum $S(k)$

$$S(k) = \frac{1}{N}\sum_{1}^{N} s(n)e^{\frac{-i2\pi nk}{N}} \qquad (23)$$

where k is the number of the harmonic frequencies.

When the oscillation response is linear (i.e., both strain and stress waveforms are sinusoidal functions), then only the fundamental frequency (k = 1) contributes to the response, and the equation is applied to the discrete signals of strain and stress. The real and imaginary parts of complex strain and complex stress can be obtained:

$$\gamma' = \frac{1}{N}\sum_{1}^{N} \gamma_n \cos\left(\frac{2\pi n}{N}\right),\ \gamma'' = \frac{1}{N}\sum_{1}^{N} \gamma_n \sin\left(\frac{2\pi n}{N}\right) \qquad (24)$$

$$\sigma' = \frac{1}{N}\sum_{1}^{N} \sigma_n \cos\left(\frac{2\pi n}{N}\right),\ \sigma'' = \frac{1}{N}\sum_{1}^{N} \sigma_n \sin\left(\frac{2\pi n}{N}\right) \qquad (25)$$

The magnitude of the complex strain and stress vectors, as well as the phase shift, are derived as Equations 26 and 27, and shown in Figure 6.

$$|\gamma^*| = (\gamma'^2 + \gamma''^2)^{1/2},\ \delta_\gamma = \tan^{-1}(\gamma''/\gamma') \qquad (26)$$

$$|\sigma^*| = (\sigma'^2 + \sigma''^2)^{1/2},\ \delta_\sigma = \tan^{-1}(\sigma''/\sigma') \qquad (27)$$

The complex modulus ($|G^*|$) and phase shift (δ) can be obtained as:

$$\delta = \delta_\sigma - \delta_\gamma \qquad (28)$$

$$|G^*| = |\sigma^*| / |\gamma^*| \qquad (29)$$

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ACKNOWLEDGMENTS

For more information or to request a product quote, please visit www.tainstruments.com to locate your local sales office information.

This paper was written by Ryan Li, PhD.