Evaluation of the correct Modulus in Rectangular Torsion

Aly Franck, TA Instruments, Germany

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SCOPE

Torsion measurements are performed extensively on rotational rheometers to characterize the low temperature behavior of polymers, composites, blends etc. The typical sample shape is rectangular as the test specimen can be easily cut from a molded or extruded sheet of material. Due to the flat aspect ratio, the test specimen can be cooled or heated up with 3^{o}C/min and more, depending on the sample thickness without any major temperature gradients throughout the sample. Glass transitions as well as secondary transitions can be measured fast and accuratly by ramping the temperature. However torsion measurements on rectangular samples do overestimate the shear modulus, especially when short samples are being used.

In the following the measurement of the modulus in torsion rectangular is investigated, the origin of the errors analyzed and a path forward to correct these errors is presented.

INTRODUCTION

For the rheologist, the easiest approach to investigate torsion is to consider a disc shaped sample, used to do shear measurements in rotational rheometers. In order to relate the material parameters strain (γ) and stress (σ) to the instrument parameters rotation angle (θ) and torque (M) two geometry constants, the strain and stress constants are used. For parallel plates, these constants are:

$$K_\gamma = \frac{\gamma}{\theta} = \frac{R}{h} \qquad K_\sigma = \frac{\sigma}{M} = \frac{2}{\pi R^3} \tag{1}$$

R is the maximum radius of the disk and h is the thickness. Note that both constants depend on the radius, which means that the stress and the strain are function of the radius i.e. the strain is zero in the center and has a maximum value at the edge of the disk.

If the same disc is extended to a long cylinder or rod, the disk radius R is also the radius R of the rod, and the thickness h becomes the length l of the rod (Figure 1).

For parallel plates, the angular displacement θ is used to describe the deformation, in torsion the angular displacement referred to the cylinder length or the twist D is used :

$$D = \frac{\theta}{l} \tag{2}$$

The modulus of the material can be related to the twist and the applied torque such as:

$$G = \frac{\tau}{\gamma} = \frac{K_\sigma M}{K_\gamma \theta} = K_g \frac{M}{\theta} = \frac{K_g}{l}\frac{Ml}{\theta} = \frac{Ml}{J_t \theta} \quad (3)$$

$$\text{with } J_t = \frac{l}{K_g} = \frac{\pi R^4}{2}$$

J_t is the polar moment of inertia and assumes the role of the torsion geometry constant for cylindrical samples. The modulus multiplied with this geometry constant GJ_t is referred to as torsional stiffness. K_g is the ratio of the stress and strain constants.

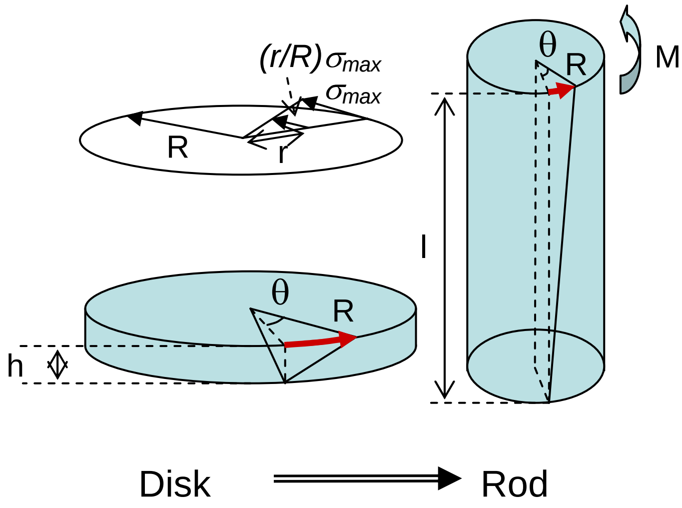

Figure 1: Torsion of a cylindrical shaped test specimen

When a cylindrical shaped test specimen is subjected to a twist or torsional deformation, disk shaped layers of solid material are rotated past each other and the only stress is the shear stress. No stresses in normal direction, along the central rotation axes are present. As such the torsion of a cylindrical shaped sample is no different to shearing a disk shaped sample between parallel plates. Since the deformations in solid materials are small, the material behaves linear and no special correction need to be applied in order to obtain the correct shear modulus.

TORSION OF A RECTANGULAR SHAPED TEST SPECIMEN

When a non symmetric test specimen such as a rectangular shaped test specimen is used, the analysis becomes much more complex. For a sample with cylindrical cross sections the stress is, due to the symmetry, the same at equal distance from the rotation axes. In a rectangular shaped sample, the constant stress lines are not linear and symmetric to the center of rotation (Figure 2). The maximum stress line is the centerline of the long side of the rectangle.

At the four corners of the rectangle, the stresses are zero. For a rectangular bar the two cross section faces are not flat, but warped due to the non linear stress distribution, shown in the three dimensional representation in figure 2. De Saint Vénant was the first to describe the torsion of a non symmetric test specimen. The polar moment of inertia for a rectangular sample is given as follows[1]:

$$J_t = \frac{t^3 w}{3} - \frac{64 t^4}{\pi^5} \sum_{n=0}^{\infty} \frac{1}{2(n+1)^5} \tanh \frac{(2n+1)\pi w}{2n} \quad \text{for } w \geq t \quad (4)$$

t is the thickness and w the width of the rectangular shaped test specimen. Since this series converges fast, higher terms can be neglected and the expression for the modulus G can be approximated by:

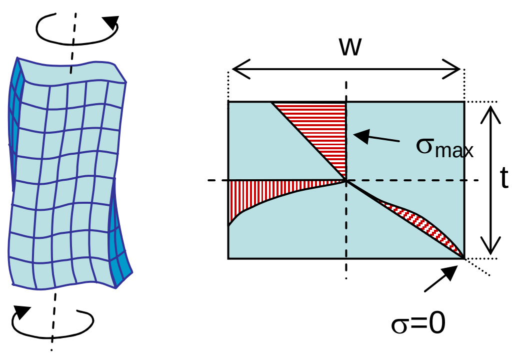

Figure 2: Stress distribution for a rectangular shaped

$$G = \frac{M}{DJ_t} = \frac{M}{\theta} \frac{3l_{nom}}{wt^3 g_{SV}(u)}$$

$$\text{with } g_{SV}(u) \cong 1 - \frac{192}{\pi^5}\frac{1}{u}\left[\tanh\frac{\pi}{2}u + 0.004524\right]$$

$$u = w/t \qquad (5)$$

l_{nom} is the nominal length between the clamps of the torsion fixture.

Because of the non linear stress distribution the sample warps at the cross section faces. In the rheometer however the deformation at the sample ends is restricted by the sample clamps. When the sample is twisted, tension stresses arise in the outer layer of the rectangular sample at the on-set of the clamps and compression stresses appear in the center. The projections of these stresses into the cross section plane of the bar alter the torque associated to the shear deformation. If the tension stresses dominate, the sample would contract if not rigidly held by the clamps.

Tension and compression stresses appear at the on-set of the clamps because the clamps hinder the test specimen to warp at the cross section faces. This leads to an increase of the torsional stiffness; a too high shear modulus is measured.
Szabo[1] suggested to correct for these effects by replacing the nominal sample length l_{nom} with a corrected sample length l_{corr}:

(*)

$$l_{corr} = l_{nom} - \kappa w \quad \text{with} \quad \kappa(u,\nu) \qquad (6)$$

ν is the Poisson ratio and $u=w/t$ the aspect ratio (width/thickness).

The length correction varies with the aspect ratio and for u>10, κ is independent of u and approaches a value of ~0.65 for a Poisson ratio of 0.33 (Figure 3).

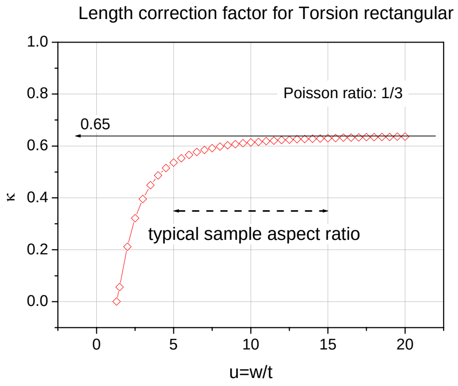

Figure 3: κ as a function of the aspect ration u=w/t. Typical rectangular samples have an aspect ratio in the range 5<u<15. A value of 0.63 for κ is a good approximation under most testing conditions

CORRECTIONS FOR TORSION RECTANGULAR IN THE ARES RHEOMETER.

Validation of the equations used in the rheometer software

The ARES rheometer measures independently the torque and the angular displacement and uses the following strain and stress constants to derive the stress and strain[2]:

$$K_\gamma = \frac{\gamma}{\theta} = \frac{R}{l_{nom}}\left[1 - 0.378\left(\frac{t}{w}\right)^2\right]$$

$$K_\sigma = \frac{\sigma}{M} = \frac{3 + 1.8\left(\frac{t}{w}\right)}{wt^2} \qquad (7)$$

The geometry constant K_g to convert torque and angular displacement to shear modulus can be reduced to:

(*)

$$\kappa = \frac{(1+\nu)(1+\nu^2)}{250}\left[25(4\frac{1-\nu^2}{1+\nu^2} - 1)k - 5(1-\nu)(1+\nu^{2^3})k^3 - 2(1+\nu)\nu^2 k^5\right]$$

$$k^2 = -\frac{3}{4}(1+u^2) + \sqrt{\left[\frac{3}{4}(1+u^2)\right]^2 + \frac{5}{2(1+\nu)}\left[4\frac{u^2-1}{1+\nu^2} - u^2\right]} \quad \text{with} \quad \nu = 1/u; u = w/t$$

$$K_g = \frac{K_\sigma}{K_\gamma}$$

$$K_g = \frac{3l_{nom}}{wt^3 g(u)} \quad \text{with} \quad g(u) = \frac{1-0.378u^{-2}}{1+0.6u^{-1}} \qquad (8)$$

The parameters *u=w/t and v=t/w* are the aspect ratio and its inverse.

In order to determine whether the equations in the rheometer calculate the modulus according to the De Saint Vénant predictions or not, the function $g_{SV}(u)$ derived from equation 5 and the function g(u) from the equations used in the ARES rheometer were compared by Dirk Wötzel[3]. The ratio of $g(u)/g_{SV}(u)$ as a function of u is shown in figure 4.

For all aspect ratios u>1.45 $g(u)/g_{SV}(u)$ is equal to 1:this means that g(u) describes the predictions of De Saint Vénant well and the equations used in the rheometer are valid.

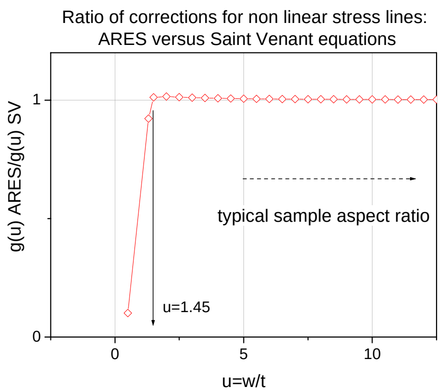

Figure 4: $g_{SV}(u)$ according to the De Saint Vénant compared g(u) from to rheometer equations

Analysis of the length correction in the torsion rectangular for nominal sample lengths from 10 to 40mm

Dirk Wötzel[3] also evaluated the shear moduli for different aspect ratios and different lengths using the traditional torsion fixture on the RDA2 rheometer. He found, that the length corrections according to Szabo[1] (equation 6) did not give the correct shear modulus when the nominal sample length was < 30mm. He extended the equation for the sample length correction and coupled the correction to the ratio of sample width and nominal length

$$l_{corr} = l_{nom}\left[1-\kappa\frac{w}{l_{nom}}+0.12886\left(\frac{w}{l_{nom}}\right)^2\right]$$

$$\text{for} \quad 10 < l_{nom} < 30\,mm \qquad (9)$$

This new equation predicts the correct modulus in the range of a sample length from 10 to 30mm. Together with the correction from Szabo, the shear modulus could be correctly determined for a sample length from 10 to 80mm with the RDA2.

Experimental and results for torsion rectangular test in the ARES rheometer

The equations 9 and 6 derived for torsion testing were tested with the new torsion rectangular test fixture for the ARES (Figure 5).The oven for the ARES rheometer has been reduced in size to improve performance and liquid nitrogen consumption. The maximum sample length in the ARES oven is limited to a maximum of 40mm nominal length. In addition there is a need to work with short samples, when only a limited

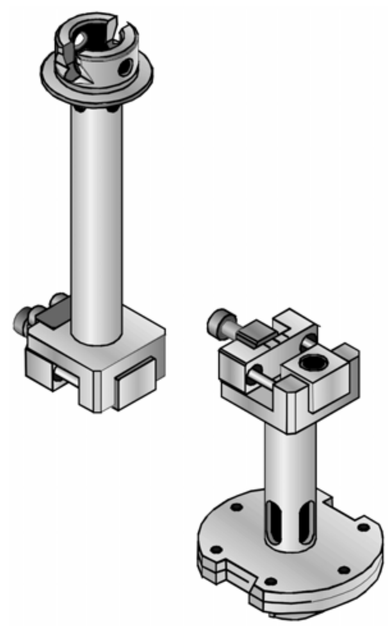

Figure 5: New torsion rectangular fixture for ARES and ARES-G2

amount of sample is available.

ABS samples with an aspect ratio u=4 (w=12.7mm;t=3.15mm) were tested at 27°C in a dynamic time sweep. The applied frequency was 1 rad/s and the strain 0.05%. Samples with a nominal length from 10 to 60mm were tested.

Figure 6 shows G' and tan δ for a typical test, obtained over a period of 200 seconds in a time sweep. The shear modulus used for the calculations was averaged over 10 measurement points. The results are listed in table 1 for all the different sample lengths investigated.

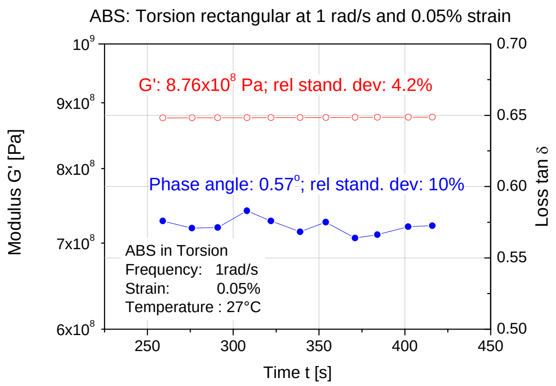

Figure 6: Measurement of the average shear modulus in the torsion rectangular test fixture

The relative standard deviation of the measured storage modulus G' is between 4 to 5%. The phase angles are in the order of 0.6°, which means that the loss modulus contribution is small and the complex shear modulus about the same than the storage modulus: G*~ G'.

The measured strain in all experiments is smaller than the command strain. The ratio of measured to command strain is 0.79 for the 10mm long sample and 0.98 for the 60mm long sample. Compliance of the test fixtures therefore needs to be corrected for. The compliance of the new test fixture was determined using the ARES with a spring transducer. The clamps were rigidly connected with a 2mm steel bar of zero length.

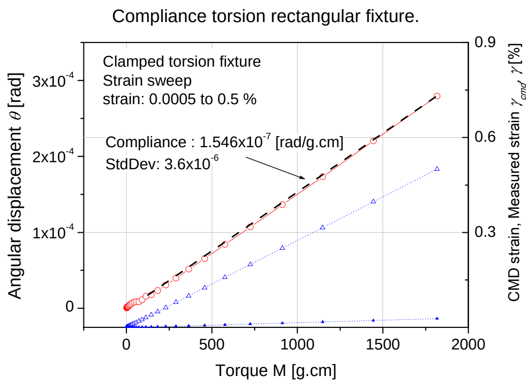

Figure 7: Compliance determination of the new Torsion Rectangular fixture using a zero lengthy steel sample

The measured shear modulus has been corrected for, using the length correction equations for short (<30mm) and long (>25mm) samples. The results are dis-

Table 1: Test results showing G' for a nominal sample length from 10 to 60mm

l_{nom}(mm)	G'measured	StdDev G'	κ	G'cor >30	G'cor >10	difference
10.6	1.30E+09	3%	4.89E-01		7.80E+08	82.81%
15.6	9.94E+08	4.5%	4.89E-01		6.84E+08	39.78%
20.6	8.68E+08	1.7%	4.89E-01		6.49E+08	22.06%
25.6	9.64E+08	4.9%	4.89E-01	7.31E+08	7.61E+08	35.56%
30.6	8.76E+08	4.6%	4.89E-01	6.99E+08	7.18E+08	23.19%
35.5	8.19E+08	4.9%	4.89E-01	6.76E+08		15.17%
40.6	8.68E+08	4,6%	4.89E-01	7.36E+08		22.06%
60.6	7.54E+08	1.4%	4.89E-01	6.77E+08		6.03%
			Average:	7.04E+08	7.18E+08	

Poisson ratio ν=1/3; aspect ratio u=w/t=4 played in table 1 and plotted in figure 8. The average corrected shear modulus over the range of the nominal sample length tested $(10<l_{nom}<60)$ is $7.11x10^8$ Pa with a relative standard deviation of 5.8%.

The difference between the corrected and uncorrected shear moduli for the 10mm long sample is ~80%, for the 60mm sample ~6% (table 1)

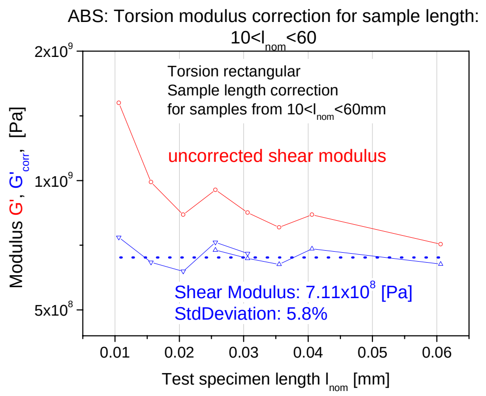

Figure 8: Torsion shear modulus uncorrected and corrected for clamping effects as a function of the nominal sample length.

CONCLUSIONS

The calculation of the shear modulus in torsion measurements, using a rectangular shaped samples has been analyzed for the torsion rectangular fixture for the ARES rheometer.

Due to the clamping of the sample, the sample cannot warp as described by the De Saint Vénant predictions. In order to compensate for these clamping effects, a length correction is applied according to Szabo. This correction can be applied for samples length for a nominal sample length from 30mm upwards. For shorter samples, a modified length correction, has been introduced. This new correction allows the prediction of the correct modulus over the sample range from 10mm to 30mm. With the two length correction formulas, the correct shear modulus in torsion can be evaluated over a range of nominal sample lengths from 10 to 60mm within 5%.

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